Effective-eigenvalue approach to the nonlinear Langevin equation for the Brownian motion in a tilted periodic potential: Application to the Josephson tunneling junction

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(Received 3 February 1993)

The effective-eigenvalue method is used for a calculation of the impedance of the Josephson tunneling junction for an externally applied small-signal alternating current in the presence of noise. The accuracy of the method is demonstrated by comparing the exact and approximate calculations. It shows clearly that the effective-eigenvalue method yields a simple and concise analytical description of the solution of the problem under consideration.

PACS number(s): 05.40.+j, 74.50.+r

I. INTRODUCTION

The theoretical description of the effects of thermal fluctuations in superconducting weak links has been developed by constructing the Fokker-Planck equation for the distribution function of the phase by analogy with the problem of the Brownian motion of a particle in a tilted periodic potential. The model has been applied for both the dc and ac Josephson effects and for the driven Josephson oscillator [1–5]. A comprehensive discussion of the Josephson junction is given in Refs. [6] and [7] and in the papers cited therein. As is well known, Josephson junction devices are very sensitive to microwave, millimeter-wave, and far-infrared signals [8]. The calculation of the response of the Josephson devices to a radio signal is generally referred to as the problem of calculating the junction impedance [9]. A knowledge of the impedance of the junction is of importance in matching the junction to the external high-frequency current [8].

We present here exact and approximate calculations of the Josephson junction impedance to an external small-signal current using the Langevin equation in the zero capacitance (noninertial) limit. A Langevin equation of the kind used for the Josephson junction problem also arises in a number of other physical situations: quantum noise in the ring-laser gyroscope [10–12], self-locking in a laser [13], the laser with injected signal [14], the theory of phase-locking techniques in radio engineering [15], etc.

The exact calculation of the linear junction impedance based on a numerical solution of the infinite hierarchy of the differential-difference equations obtained from the Fokker-Planck equation has been given in Ref. [16] (see also the discussion of these results in Ref. [17]). Another method of exact solution of this equation for a similar problem (ring-laser gyroscope) has been suggested by Cresser *et al.* [11] in terms of an infinite continued fraction (see also [6]). However, these numerical approaches to the problem have the disadvantage that they do not yield a closed-form solution. Furthermore the qualitative behavior is not at all obvious [10].

Several investigators (see, e.g., [2,17]) have attempted to overcome this problem by deriving approximate analytical expressions for various ranges of the junction parameter values. However, a general equation which would be valid for all the parameter ranges of interest has not yet been derived.

In the present paper such an approximate solution is obtained with the help of the effective-eigenvalue method which is described in detail in Ref. [18]. In our context the method constitutes a truncation procedure which allows us to obtain a closed-form approximation to the solution of the infinite hierarchy of differential-difference equations obtained directly from the Langevin equation without recourse to the Fokker-Planck equation. These equations govern the time behavior of the statistical averages characterizing the dynamics of the Josephson junction in the presence of noise. We show that the effective-eigenvalue method is a valuable and extremely powerful tool for the purpose of obtaining a simple analytical solution for the impedance of the Josephson junction. The solution obtained from the effective-eigenvalue method is shown to agree closely with the exact solution for a wide range of the bias and barrier-height parameters. This is demonstrated by the plots of the impedance as a function of these parameters. We remark that our method of obtaining the exact solution of the problem also has the merit of being considerably simpler than the previously available algorithm [16].

II. CURRENT BALANCE EQUATION FOR THE JOSEPHSON JUNCTION

The Josephson tunneling junction is made up of two superconductors separated from each other by a thin layer of oxide [7]. We label [7] ψ_R and ψ_L the wave functions for the right and left superconductors, respectively. The phase difference $\phi=\phi_R-\phi_L$, where ϕ_R and ϕ_L are the phase angles associated with the wave functions ψ_R and ψ_L, respectively, is given by the Josephson equation [6,7]

$$\dot{\phi}(t)=\frac{2eV(t)}{\hbar}, \tag{1}$$

where $V(t)$ is the potential difference across the junction; e is the charge on the electron; and $\hbar=h/2\pi$, where h is the Planck constant. The junction is now modeled (Fig. 1) [6,7] by a resistor R in parallel with a capacitance C across which is connected a dc current generator $I_{\rm dc}$ (representing the bias current applied to the junction). At the other end of the junction (across the resistor R) is connected a phase-dependent current generator, $I\sin\phi$, representing the Josephson supercurrent due to the Cooper pairs tunneling through the junction. Since the junction operates at temperatures above absolute zero, there is a noise current $L(t)$ superimposed on the bias current satisfying the conditions

$$\overline{L(t_1)L(t_2)}=\frac{2kT}{T}\delta(t_1-t_2),\quad \overline{L(t)}=0\ . \tag{2}$$

The overbar denotes "the statistical average of," $\delta(t)$ is the Dirac delta function, and t_1 and t_2 are distinct times.

The current balance equation for the junction is [6,7]

$$I_{\rm dc}+L(t)=C\frac{dV(t)}{dt}+\frac{V(t)}{R}+I\sin[\phi(t)]\ . \tag{3}$$

Substitution of Eq. (1) in Eq. (3) yields

$$\left[\frac{\hbar}{2e}\right]^2 C\ddot{\phi}(t)+\frac{1}{R}\left[\frac{\hbar}{2e}\right]^2\dot{\phi}(t)+\left[\frac{\hbar}{2e}\right]I\sin[\phi(t)] =\frac{\hbar}{2e}\left[I_{\rm dc}+L(t)\right]\ . \tag{4}$$

Equation (4) has the same form as that for a Brownian particle of mass $(\hbar/2e)^2C$ moving in the tilted cosine potential [6]

$$U(\phi)=-\frac{\hbar}{2e}(I_{\rm dc}\phi+I\cos\phi)\ . \tag{5}$$

This is the Langevin equation of the Josephson junction. In general, Eq. (4) can be solved by the methods described by Risken [6]. However, we shall consider Eq. (4) in the diffusion (noninertial or low frequency) limit only

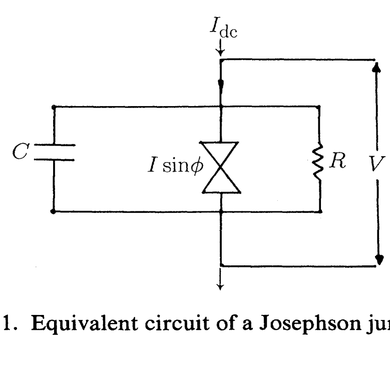

FIG. 1. Equivalent circuit of a Josephson junction.

where we can neglect the capacitative term $C\ddot{\phi}(t)$. Thus the Langevin equation becomes [19]

$$\frac{d}{dt}\phi(t)+\frac{2eR}{\hbar}I\sin[\phi(t)]=\frac{2eR}{\hbar}[I_{\rm dc}+L(t)]\ . \tag{6}$$

III. REDUCTION OF THE AVERAGED LANGEVIN EQUATION TO A SET OF DIFFERENTIAL-DIFFERENCE EQUATIONS

In order to proceed we change the variable in the Langevin equation (6) by means of the transformation of the variables

$$r^n=e^{-in\phi}\quad (n=\ldots,-1,0,1,\ldots)$$

so that

$$\frac{d}{dt}r^n(t)=\frac{enIR}{\hbar}[r^{n-1}(t)-r^{n+1}(t)] -\frac{i2enR}{\hbar}r^n(t)[I_{\rm dc}+L(t)]\ . \tag{7}$$

The multiplicative noise term $r^n(t)L(t)$ in Eq. (7) contributes a noise-induced drift term to the average [6]. This term poses an interpretation problem in averaging Eq. (7). We recall that, taking the Langevin equation for a stochastic variable $\xi(t)$ as [6]

$$\frac{d}{dt}\xi(t)=h(\xi(t),t)+g(\xi(t),t)L(t) \tag{8}$$

with

$$\overline{L(t)}=0,\quad \overline{L(t)L(t')}=2\delta(t-t')\ ,$$

and interpreting it as a Stratonovich stochastic equation [6], we have

$$\dot{x}=\lim_{\tau\to 0}\left.\left\{\frac{1}{\tau}\overline{[\xi(t+\tau)-x]}\right\}\right|_{\xi(t)=x} =h(x,t)+g(x,t)\frac{\partial}{\partial x}g(x,t)\ , \tag{9}$$

where $\xi(t+\tau)$, $\tau>0$ is a solution of Eq. (8) which at time t has the sharp value $\xi(t)=x$. It should be noted that the quantity x in Eq. (9) is itself a random variable with probability density function $W(x,t)$ defined such that $W(x,t)dx$ is the probability of finding x in the interval $(x,x+dx)$. Thus on averaging Eq. (9) over $W(x,t)$ we obtain

$$\frac{d}{dt}\langle x\rangle=\langle h(x,t)\rangle+\left\langle g(x,t)\frac{\partial}{\partial x}g(x,t)\right\rangle\ , \tag{10}$$

where the angular braces mean the relevant quantity averaged over $W(x,t)$.

We may use the above results to evaluate the average of the multiplicative noise term in Eq. (9). We have

$$g(r^n)=-\frac{i2enRr^n}{\hbar}\ ,\qquad g(r^n)\frac{\partial}{\partial r^n}g(r^n)=-\left[\frac{2neR}{\hbar}\right]^2 r^n\ , \tag{11}$$

and

$$\frac{d}{dt}r^n=\frac{eInR}{\hbar}(r^{n-1}-r^{n+1})-\frac{in2eR}{\hbar}I_{\rm dc}r^n -kTR^2\left[\frac{2en}{\hbar}\right]^2 r^n \ . \qquad (12)$$

Thus we obtain the hierarchy of differential-difference equations for the averages

$$\frac{d}{dt}\langle r^n\rangle+\frac{1}{\tau_0}(n^2+\frac{inx\gamma}{2})\langle r^n\rangle =\frac{n\gamma}{4\tau_0}(\langle r^{n-1}\rangle-\langle r^{n+1}\rangle)\ , \qquad (13)$$

where

$$x=\frac{I_{\rm dc}}{I} \qquad (14)$$

is the ratio of bias current amplitude to supercurrent amplitude (this is the bias or tilt parameter),

$$\gamma=\frac{\hbar I}{ekT} \qquad (15)$$

is the ratio of Josephson coupling energy to thermal energy (the barrier-height parameter), and

$$\tau_0=\left[\frac{\hbar}{2e}\right]^2\frac{1}{kTR} \qquad (16)$$

is the characteristic relaxation time.

We remark that $r^n(t)$ in Eq. (7) and r^n in Eqs. (12) and (13) have different meanings, namely, $r^n(t)$ in Eq. (7) is a stochastic variable while in Eqs. (12) and (13) r^n is the sharp (definite) value $r^n(t)=r^n$ at time t. (Instead of using different symbols for the two quantities we have distinguished the sharp values at time t from the stochastic variables by deleting the time argument as in Ref. [6]). The quantity r^n above is itself a random variable which must be averaged over an ensemble of junctions. The symbol $\langle\ \rangle$ means such an ensemble average.

Equation (13) is a well-known result [6] which may be obtained from the relevant Fokker-Planck equation. This equation in the noninertial limit is [1,6]

$$\zeta\frac{\partial W}{\partial t}=\frac{\partial}{\partial\phi}\left[W\frac{\partial}{\partial\phi}U\right]+kT\frac{\partial^2 W}{\partial\phi^2}\ , \qquad (17)$$

where $W(\phi,t)$ is the transition probability of the phase, ζ is the damping coefficient defined as

$$\zeta=\left[\frac{\hbar}{2e}\right]^2\frac{1}{R}\ ,$$

and U is the tilted cosine potential given by Eq. (5).

The distribution function W must be periodic in such a way that it can be expanded in a Fourier series as [6]

$$W(\phi,t)=\sum_{p=-\infty}^{\infty}a_p(t)e^{ip\phi}\ . \qquad (18)$$

Substituting Eq. (18) into Eq. (17) and using the orthogonality properties of the circular functions one can find that the coefficients $a_p(t)$ satisfy

$$\frac{d}{dt}a_p(t)+\frac{kT}{\zeta}\left[p^2+\frac{ip\hbar I_{\rm dc}}{2ekT}\right]a_p(t) =\frac{p\gamma kT}{4\zeta}[a_{p-1}(t)-a_{p+1}(t)]\ . \qquad (19)$$

It can be easily shown that the $a_p(t)$ of Eq. (18) are related to $\langle r^p\rangle$ by

$$a_p(t)=\frac{1}{2\pi}\langle e^{-ip\phi}\rangle=\frac{1}{2\pi}\langle r^p\rangle\ . \qquad (20)$$

Thus Eq. (19) coincides precisely with Eq. (13).

IV. EXACT SOLUTION FOR THE LINEAR RESPONSE TO AN APPLIED ALTERNATING CURRENT

We suppose that the current across the junction is now $I_{\rm dc}+I_m\exp(-i\omega t)$. We also suppose that $\hbar I_m/2ekT\ll 1$ so that we can make the perturbation expansion (we may use the exponential form for the ac current since we seek only the linear response),

$$\langle r^p\rangle=\langle r^p\rangle_0+\tilde{A}_p(\omega)I_m\exp(-i\omega t)/I+\cdots\ . \qquad (21)$$

On substituting Eq. (21) into Eq. (13) we obtain for the linear response to the ac current

$$\left[-i\omega\tau_0+p^2+\frac{ip\gamma}{2}x\right]\tilde{A}_p(\omega)=\frac{\gamma p}{4}[\tilde{A}_{p-1}(\omega)-\tilde{A}_{p+1}(\omega)] -\frac{ip\gamma}{2}\langle r^p\rangle_0 \qquad (22)$$

with $\tilde{A}_0(\omega)=0$.

Equation (22) can readily be solved for the homogeneous case, i.e., for $\langle r^p\rangle_0=0$. Denoting the solution of the homogeneous equation (22) by $\tilde{r}_p(\omega)$ and introducing the quantity $\tilde{S}_p(\omega)$ defined as

$$\tilde{S}_p(\omega)=\tilde{r}_p(\omega)/\tilde{r}_{p-1}(\omega)\ , \qquad (23)$$

we have the continued fraction solution of the homogeneous equation (22), namely [6],

$$\tilde{S}_p(\omega)=\frac{0.5}{-\frac{2i\omega\tau_0}{\gamma p}+\frac{2p}{\gamma}+ix+0.5\tilde{S}_{p+1}(\omega)}\ . \qquad (24)$$

In particular, for $p=1$ we obtain

$$\tilde{S}_1(\omega)=\cfrac{0.5}{i\left[x-\frac{2\omega\tau_0}{\gamma}\right]+\frac{2}{\gamma}+\cfrac{0.25}{i\left[x-\frac{2\omega\tau_0}{2\gamma}\right]+\frac{4}{\gamma}+\cfrac{0.25}{i\left[x-\frac{2\omega\tau_0}{3\gamma}\right]+\cdots}}}\ . \qquad (25)$$

It can be seen that from Eq. (24)

$$\tilde{S}_p(0)=S_p=\frac{\langle r^p\rangle_0}{\langle r^{p-1}\rangle_0} \tag{26}$$

is the stationary solution of the averaged Langevin equation (13).

As shown in Appendix A we can now solve the inhomogeneous equation (22) by successively eliminating the variables to obtain $\tilde{A}_1(\omega)$ and $\tilde{A}_{-1}(\omega)$ in terms of $\tilde{S}_1(\omega)$ [Eqs. (A18) and (A22), respectively]. Having determined $\tilde{A}_1(\omega)$ and $\tilde{A}_{-1}$ we can calculate $\langle \sin\phi-\langle \sin\phi\rangle_0\rangle$, namely,

$$\langle \sin\phi-\langle \sin\phi\rangle_0\rangle=D_s(\omega)\frac{I_m e^{-i\omega t}}{I}, \tag{27}$$

where

$$D_s(\omega)=\frac{i}{2}[\tilde{A}_1(\omega)-\tilde{A}_{-1}(\omega)]$$
$$=\frac{i\gamma}{4\tau_0\omega}[S_1+S_1^*-\tilde{S}_1(\omega)-\tilde{S}_1^*(-\omega)] \tag{28}$$

and the asterisk means complex conjugate.

Furthermore, one may use Eq. (28) to evaluate the impedance $Z(\omega)$ of the junction. In order to accomplish this we recall that the averaged current balance equation in the presence of the ac is

$$I_{dc}+I_m e^{-i\omega t}-I\langle \sin\phi\rangle-\frac{\langle V\rangle}{R}=0 . \tag{29}$$

We have supposed that

$$\langle \sin\phi\rangle=\langle \sin\phi\rangle_0+\langle \sin\phi\rangle_1 ,$$
$$\langle V\rangle=\langle V\rangle_0+\langle V\rangle_1 ,$$

where the subscript 0 on the angular braces denotes the average in the absence of the ac, and the subscript 1 the portion of the average which is linear in I_m. Thus on noting that

$$\frac{I_{dc}}{I}-\langle \sin\phi\rangle_0-\frac{\langle V\rangle_0}{IR}=0$$

and using Eqs. (27) and (28) we have

$$\langle V\rangle_1=Z(\omega)I_m e^{-i\omega t} , \tag{30}$$

where $Z(\omega)$ is the impedance of the junction given by

$$Z(\omega)=R[1-D_s(\omega)]$$
$$=\left\{1-\frac{i\gamma}{4\tau_0\omega}[S_1+S_1^*-\tilde{S}_1(\omega)-\tilde{S}_1^*(-\omega)]\right\} . \tag{31}$$

V. THE EFFECTIVE EIGENVALUES AND THE LINEAR RESPONSE OF THE JOSEPHSON JUNCTION

Let us now suppose that a strong dc current I_{dc} had been applied to the junction in the infinite past and that at $t=0$, I_{dc} is incremented by a small current $U(t)\Delta$, where $U(t)$ is the unit step function so that the total current is $I_{dc}+U(t)\Delta$. Now we are only interested in the response linear in Δ. We therefore assume that

$$\langle r^n\rangle=\langle r^n\rangle_{eq}+\langle r^n\rangle_1 , \tag{32}$$

where the subscript 1 denotes the portion of the statistical average which is linear in Δ and the subscript eq denotes the statistical average in the stationary state computed using the stationary distribution function [6]

$$W_0(\phi)=C_0e^{-U(\phi)/kT}\times\left[1-\frac{(1-e^{-U(\phi)/kT})\int_0^{\phi}e^{U(\phi')/kT}d\phi'}{\int_0^{2\pi}e^{-U(\phi')/kT}d\phi'}\right] , \tag{33}$$

where

$$U(\phi)=-\hbar[\cos\phi+(I_{dc}+\Delta)\phi]/2e . \tag{34}$$

On substituting Eq. (32) into Eq. (13) we obtain

$$\left[n^2+\frac{in\hbar}{2ekT}(i_{dc}+\Delta)\right]\langle r^n\rangle_{eq}+\frac{n\hbar I}{4ekT}(\langle r^{n+1}\rangle_{eq}-\langle r^{n-1}\rangle_{eq})=0 \tag{35}$$

and

$$\frac{d}{dt}\langle r^n\rangle_1+\frac{1}{\tau_0}\left[n^2+\frac{in\hbar}{2ekT}I_{dc}\right]\langle r^n\rangle_1=\frac{n\hbar I}{4\tau_0 ekT}(\langle r^{n-1}\rangle_1-\langle r^{n+1}\rangle_1)-\frac{i\Delta n\hbar}{2e\tau_0 kT}\langle r^n\rangle_{eq}U(t) . \tag{36}$$

Equation (36) is a three-term recurrence relation driven by a forcing function, namely, the $U(t)$ term. In order to determine the effective eigenvalue for the quantity of interest which is $\langle r\rangle_1$ we shall consider the unforced part of Eq. (36) at $n=1$ and reduce it to an eigenvalue problem

$$\frac{d}{dt}\langle r\rangle_1+\lambda_{ef}^+\langle r\rangle_1=0 , \tag{37}$$

where λ_{ef} is the effective eigenvalue to be determined. Since $\langle r\rangle_1$ is a complex variable λ_{ef} is also complex, namely,

$$\lambda_{ef}^+=\lambda=\lambda'+i\lambda'' . \tag{38}$$

The real part of λ_{ef}^+ when inverted will give the effective relaxation time while the imaginary part will give the frequency of oscillation.

For any time t, Eq. (37) yields

$$\lambda_{ef}^+=-\frac{\frac{d}{dt}\langle r\rangle_1}{\langle r\rangle_1} . \tag{39}$$

The effective-eigenvalue method [18] suggests that Eq. (39) may be replaced by its initial value (i.e., its value at $t=0$). We therefore have

$$\lambda_{\rm ef}^{+}=-\left.\frac{\frac{d}{dt}\langle r\rangle_1}{\langle r\rangle_1}\right|_{t=0} . \tag{40}$$

On substituting Eq. (36) into Eq. (40) for $n=1$ we obtain

$$\lambda_{\rm ef}^{+}=\frac{1}{\tau_0}+\frac{i2eR}{\hbar}I_{\rm dc}+\frac{\langle r^2\rangle_1}{\langle r\rangle_1}\frac{eIR}{\hbar} , \quad t=0 . \tag{41}$$

Now from Eq. (32) we have

$$\lim_{t\to 0}\langle r^n\rangle_1=\langle r^n\rangle_0-\langle r^n\rangle_{\rm eq} . \tag{42}$$

Equation (41) with the aid of Eq. (42) simplifies further to

$$\lambda_{\rm ef}^{+}=\frac{1}{\tau_0}+\frac{i2eR}{\hbar}I_{\rm dc}+\left[\frac{\langle r^2\rangle_{\rm eq}-\langle r^2\rangle_0}{\langle r\rangle_{\rm eq}-\langle r\rangle_0}\right]\frac{eIR}{\hbar} . \tag{43}$$

The averages $\langle r\rangle_{\rm eq}$ and $\langle r^2\rangle_{\rm eq}$ in Eq. (43) are over the stationary distribution $W_0(\phi)$ of Eq. (33) with the perturbed potential of Eq. (34). However, remembering that we are interested only in the linear response of Δ we can express $\langle r^n\rangle_{\rm eq}-\langle r^n\rangle_0$ as

$$\langle r^n\rangle_{\rm eq}-\langle r^n\rangle_0=\Delta\frac{\partial}{\partial\Delta}\langle r^n\rangle_0+O(\Delta^2) . \tag{44}$$

Therefore in order to determine $\lambda_{\rm ef}^{+}$ from Eq. (43) we only need to evaluate $(\partial/\partial\Delta)\langle r\rangle_0$ and $(\partial/\partial\Delta)\langle r^2\rangle_0$ as given in Appendix B. The final result is

$$\lambda_{\rm ef}^{+}=\frac{\gamma}{4\tau_0}\frac{\langle r\rangle_0}{\sum_{n=1}^{\infty}(-1)^{n+1}\langle r^n\rangle_0^2} . \tag{45}$$

Thus we have expressed the complex effective eigenvalue $\lambda_{\rm ef}^{+}=\lambda'+i\lambda''$ in terms of the equilibrium averages $\langle r^n\rangle_0$ only. However, for numerical calculations it is more convenient to express $\lambda_{\rm ef}^{+}$ in terms of the continued fraction S_n. On using Eq. (26) we obtain after some algebra

$$\lambda_{\rm ef}^{+}=\frac{\gamma}{4\tau_0 S_1(1-S_2^2\{1-S_3^2[1-S_4^2(1-\cdots)]\})} . \tag{46}$$

We can show in the same way that the effective eigenvalue $\lambda_{\rm ef}^{-}$ for $\langle r^{-1}\rangle$ is related to $\lambda_{\rm ef}^{+}$ by

$$\lambda_{\rm ef}^{-}=(\lambda_{\rm ef}^{+})^*=\lambda^*=\lambda'-i\lambda'' . \tag{47}$$

The behavior of the real $(\tau_0\lambda')$ and imaginary $(\tau_0\lambda'')$ parts of the normalized effective eigenvalue $\tau_0\lambda_{\rm ef}^{+}$ as a function of the barrier height γ and bias parameters x is illustrated in Figs. 2 and 3. As we shall see in Sec. VI, Eq. (46) accurately represents the behavior of the frequency of oscillation $({\rm Im}\{\lambda_{\rm ef}^{+}\})$ and the spectrum broadening $({\rm Re}\{\lambda_{\rm ef}^{+}\})$.

Having determined the effective eigenvalues $\lambda_{\rm ef}^{+}$ and $\lambda_{\rm ef}^{-}$ we may calculate the impedance of the junction as follows.

We recall that the leading members of the hierarchy of differential-difference equations are

$$\frac{d}{dt}\langle r^{-1}\rangle_1+\frac{1}{\tau_0}\left[1-\frac{i\hbar I_{\rm dc}}{2ekT}\right]\langle r^{-1}\rangle_1=\frac{\hbar I}{4ekT\tau_0}(1-\langle r^{-2}\rangle_1)+\frac{i2eR}{\hbar}I_m e^{-i\omega t}\langle r^{-1}\rangle_0 , \tag{48}$$

$$\frac{d}{dt}\langle r\rangle_1+\frac{1}{\tau_0}\left[1+\frac{i\hbar I_{\rm dc}}{2ekT}\right]\langle r\rangle_1=\frac{\hbar I}{4ekT\tau_0}(1-\langle r^{2}\rangle_1)-\frac{i2eR}{\hbar}\langle r\rangle_0 I_m e^{-i\omega t} . \tag{49}$$

Using Eqs. (26) and (40), Eq. (48) and (49) reduce to the ordinary differential equations of the first order

$$\frac{d}{dt}\langle r^{-1}\rangle_1+\lambda_{\rm ef}^{-}\langle r^{-1}\rangle_1=\frac{i\gamma S_1^*}{2\tau_0 I}I_m e^{-i\omega t} , \tag{50}$$

$$\frac{d}{dt}\langle r\rangle_1+\lambda_{\rm ef}^{+}\langle r\rangle_1=-\frac{i\gamma S_1}{2\tau_0 I}I_m e^{-i\omega t} . \tag{51}$$

The steady-state solutions of Eqs. (50) and (51) are

$$\langle r^{-1}\rangle_1=\frac{i\gamma S_1^*}{2\tau_0(\lambda_{\rm ef}^{-}-i\omega)}\frac{I_m e^{-i\omega t}}{I} , \tag{52}$$

$$\langle r\rangle_1=-\frac{i\gamma S_1}{2\tau_0(\lambda_{\rm ef}^{+}-i\omega)}\frac{I_m e^{-i\omega t}}{I} . \tag{53}$$

Thus

$$\langle \sin\phi-\langle\sin\phi\rangle_0\rangle=\frac{\langle r^{-1}\rangle_1-\langle r\rangle_1}{2i}=D_s(\omega)I_m e^{-i\omega t}/I , \tag{54}$$

where

$$D_s(\omega)=\frac{\gamma}{4\tau_0}\left[\frac{S_1^*}{\lambda'-i(\omega+\lambda'')}+\frac{S_1}{\lambda'-i(\omega-\lambda'')}\right] . \tag{55}$$

Hence the impedance of the junction $Z(\omega)$ defined by Eq. (31) is given by the simple formula

$$Z(\omega)=R\left\{1-\frac{\gamma}{4\tau_0}\left[\frac{S_1^*}{\lambda'-i(\omega+\lambda'')}+\frac{S_1}{\lambda'-i(\omega-\lambda'')}\right]\right\} , \tag{56}$$

where S_1 and $\lambda'+i\lambda''$ are given by Eqs. (B8) at $n=1$ and (46), respectively.

VI. RESULTS AND DISCUSSION

We now compare the impedance $Z(\omega)$ from the approximate equation (56) with the exact solutions [Eq. (31)]. The results of the calculations are shown in Fig. 4. It is apparent by inspection of this figure that Eq. (56)

gives perfect correspondence to the exact solution for all ranges of the bias x and barrier-height γ parameters. Thus it allows us to represent the impedance of the junction $Z(\omega)$ by the simple analytic formula of Eq. (56) which describe [20] a resonance with natural angular frequency λ''.

We remark that approximate equations for the impedance of a form similar to the above equation have been derived in Ref. [17] [Eqs. (3.11) and (3.14)]. However the equations have a narrower range of applicability than ours as they are confined on the one hand [Eq. (3.11)] to large ω and on the other hand [Eq. (3.14)] to small fluctuations.

It is of interest to compare the results we have obtained with those for the noiseless case. In the absence of noise one can calculate the impedance analytically by finding the differential impedance at the bias point [8]. The derivation and discussion of the noiseless case is given elsewhere [8,9,17]. In our notation the result is as follows: For values of the bias parameter $x \le 1$

$$\frac{Z(\omega)}{R}=\frac{\Omega(\Omega-i\sqrt{1-x^2})}{1+\Omega^2-x^2} \tag{57}$$

and for $x > 1$

$$\frac{Z(\omega)}{R}=1+\frac{\sqrt{x^2-1}}{(x+\sqrt{x^2-1})(x^2-1-\Omega^2)}+\frac{i\pi}{2(x+\sqrt{x^2-1})}\delta(\Omega-\sqrt{x^2-1}) , \tag{58}$$

where

$$\Omega=2\omega\tau_0/\gamma . \tag{59}$$

The above equations have a simple physical interpretation. If $x<1$ the junction behaves like an inductance

$$L_j=\frac{\hbar/2eI}{\sqrt{1-x^2}}$$

in parallel with the resistance R, yielding the admittance

$$Y(\omega)=1/R-1/i\omega L$$

which gives the impedance $Z(\omega)=Y^{-1}(\omega)$ from Eq. (57) [8]. If $x>1$ the impedance is entirely real with a singularity at $x_s=\sqrt{1+\Omega^2}$. This singularity vanishes in the presence of noise as is evident from Fig. 4. Such behavior is even more pronounced in Fig. 5. There we have plot-

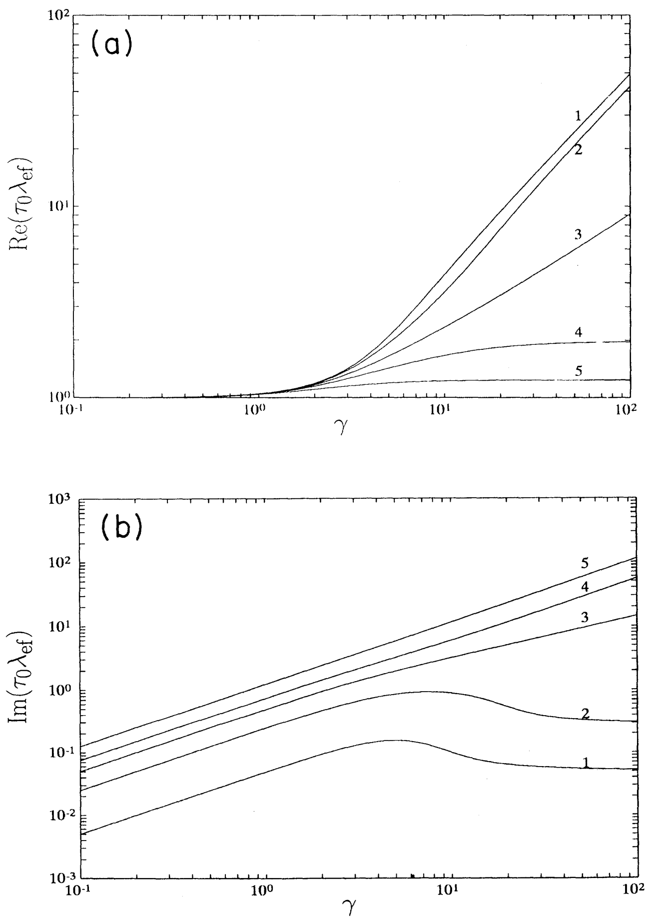

FIG. 2. The real (a) $\tau_0\lambda'$ and imaginary (b) $\tau_0\lambda''$ parts of the effective eigenvalue for the Josephson junction vs γ. Curve 1, x (bias current)$=0.1$; 2, $x=0.5$; 3, $x=1.0$; 4, $x=1.5$; and 5, $x=2.5$.

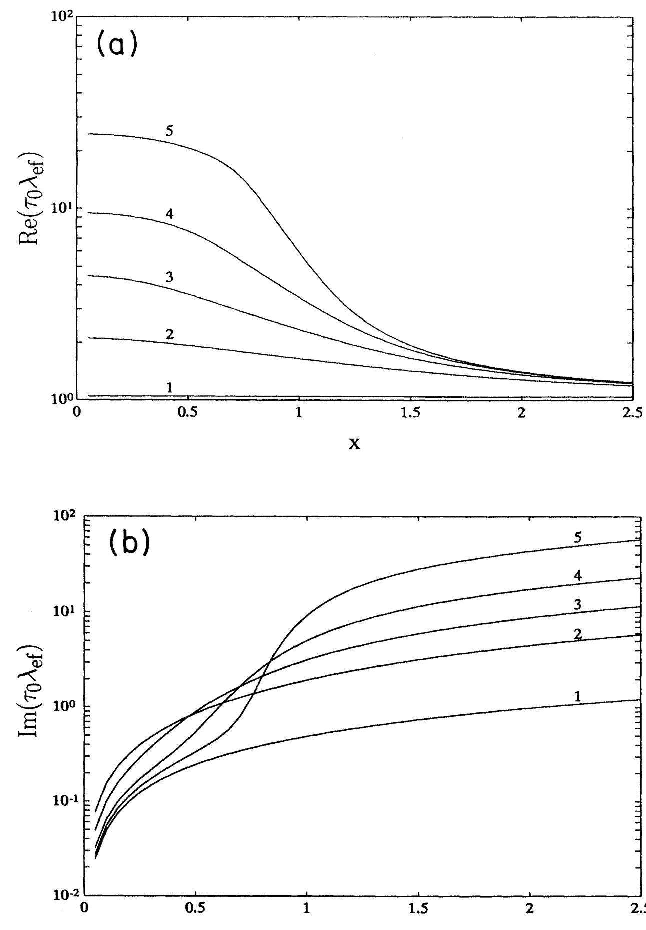

FIG. 3. The real (a) $\tau_0\lambda'$ and imaginary (b) $\tau_0\lambda''$ parts vs x. Curve 1, $\gamma=1$; 2, $\gamma=5$; 3, $\gamma=10$; 4, $\gamma=20$; and 5, $\gamma=50$.

ted the normalized impedance $Z(\omega)/R$ as a function of the bias parameter x and have compared it with the noiseless case. It is apparent from Fig. 5 that for weak noise ($\gamma=50$) Eqs. (57) and (58) yield a satisfactory description of the impedance excluding the region in the vicinity of the singular point $x_s=\sqrt{1+\Omega^2}$. However, we remark that at moderate values of γ (e.g., at $\gamma=10$), there is a striking difference between the solutions with and without noise. This is particularly apparent in Figs. 5(a) and 5(b) where close to the singular point x_s the noiseless solution (in contrast to that including noise) possesses a *negative* real part which is an indication of

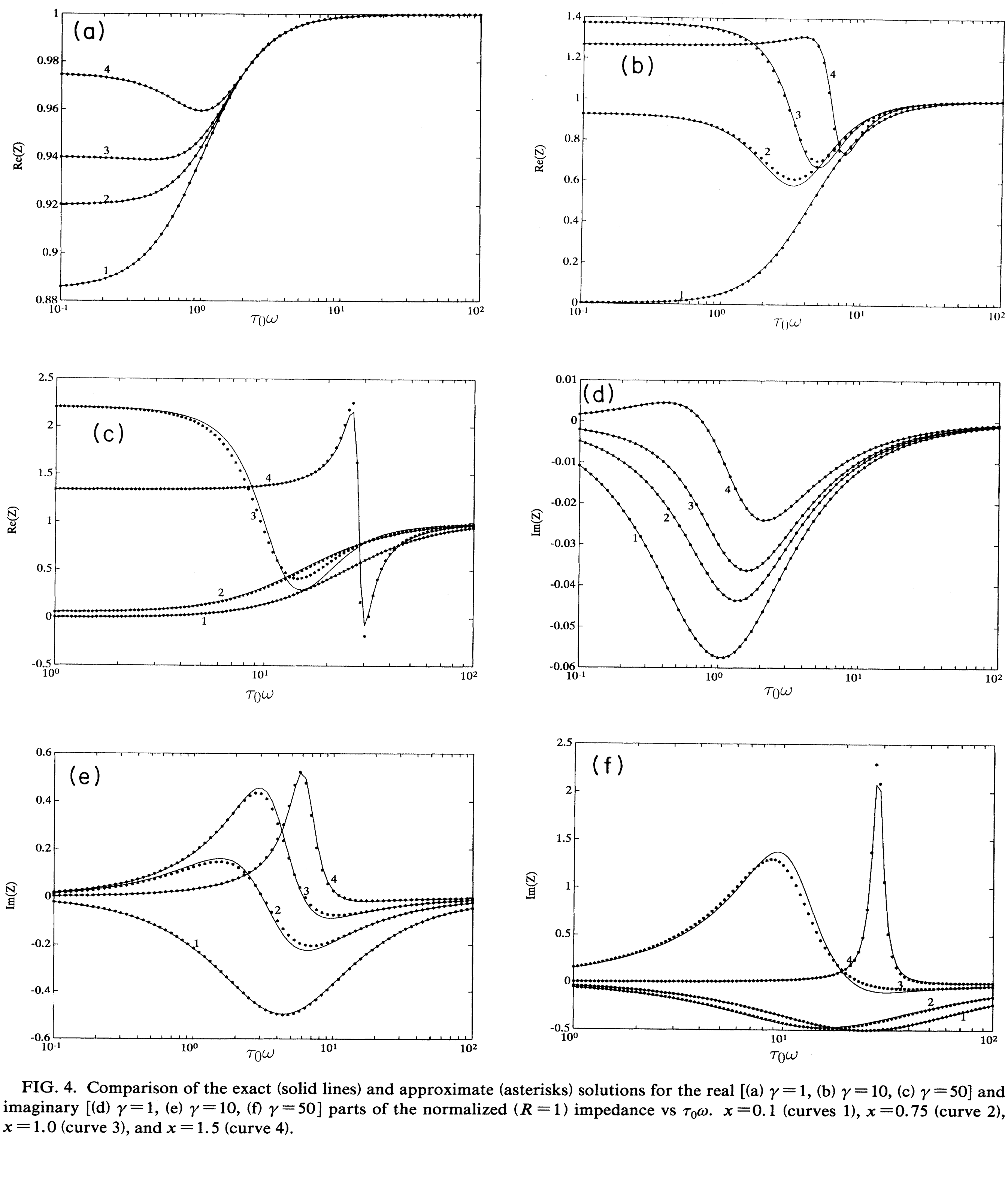

FIG. 4. Comparison of the exact (solid lines) and approximate (asterisks) solutions for the real [(a) $\gamma=1$, (b) $\gamma=10$, (c) $\gamma=50$] and imaginary [(d) $\gamma=1$, (e) $\gamma=10$, (f) $\gamma=50$] parts of the normalized ($R=1$) impedance vs $\tau_0\omega$. $x=0.1$ (curves 1), $x=0.75$ (curve 2), $x=1.0$ (curve 3), and $x=1.5$ (curve 4).

amplification or oscillation which may occur if the junction is inserted in an appropriate microwave circuit [9].

VII. CONCLUSIONS

The purpose of this paper is to demonstrate how the effective-eigenvalue method allied with the Langevin equation may be applied with much success to the Brownian motion in a tilted cosine potential. On applying the effective-eigenvalue technique to this model we find that it yields a simple analytic formula, Eq. (56), for the response of a Josephson junction to a weak ac current which agrees closely with the exact solution. This has the merit that we now have a simple analytic formula for the impedance in the presence of noise, the spectrum broadening, and the resonant frequency for all ranges of the parameters x and γ. The inclusion of noise has a profound effect on the operation of the junction as it may remove [3,4] the resonance singularity of the noiseless case. We show in addition to the derivation of these approximate formulas that the exact linear response of the Josephson junction to a weak ac current is determined by the continued fraction $\tilde{S}_1(\omega)$. This representation of the exact solution has the advantage that it can easily be adapted for iteration in order to determine the nonlinear response of the Josephson junction.

As we already mentioned in the introduction a Langevin equation of the kind used in the present paper also arises in a number of other [6] applications. Therefore the results obtained for the Josephson junction can also be applied to analogous systems, for example, to the ring-laser gyroscope [10–12]. The Langevin equation for the Josephson junction has the same mathematical form as that of the ring-laser gyroscope with an appropriate change of parameters. In the ring-laser gyroscope operating at steady state the quantity of interest is the spectrum of the beat signal (see, for example, Ref. [10]). As we have shown in Ref. [18] the effective-eigenvalue method when applied to the ring-laser gyroscope gives also a good quantitative description of the main features of the spectrum in all regions of interest [10–12].

APPENDIX A

Here we give the solution of the inhomogeneous equation (22) which governs the exact solution as follows. Let us divide Eq. (22) by $\tilde{A}_{p-1}(\omega)$ and write

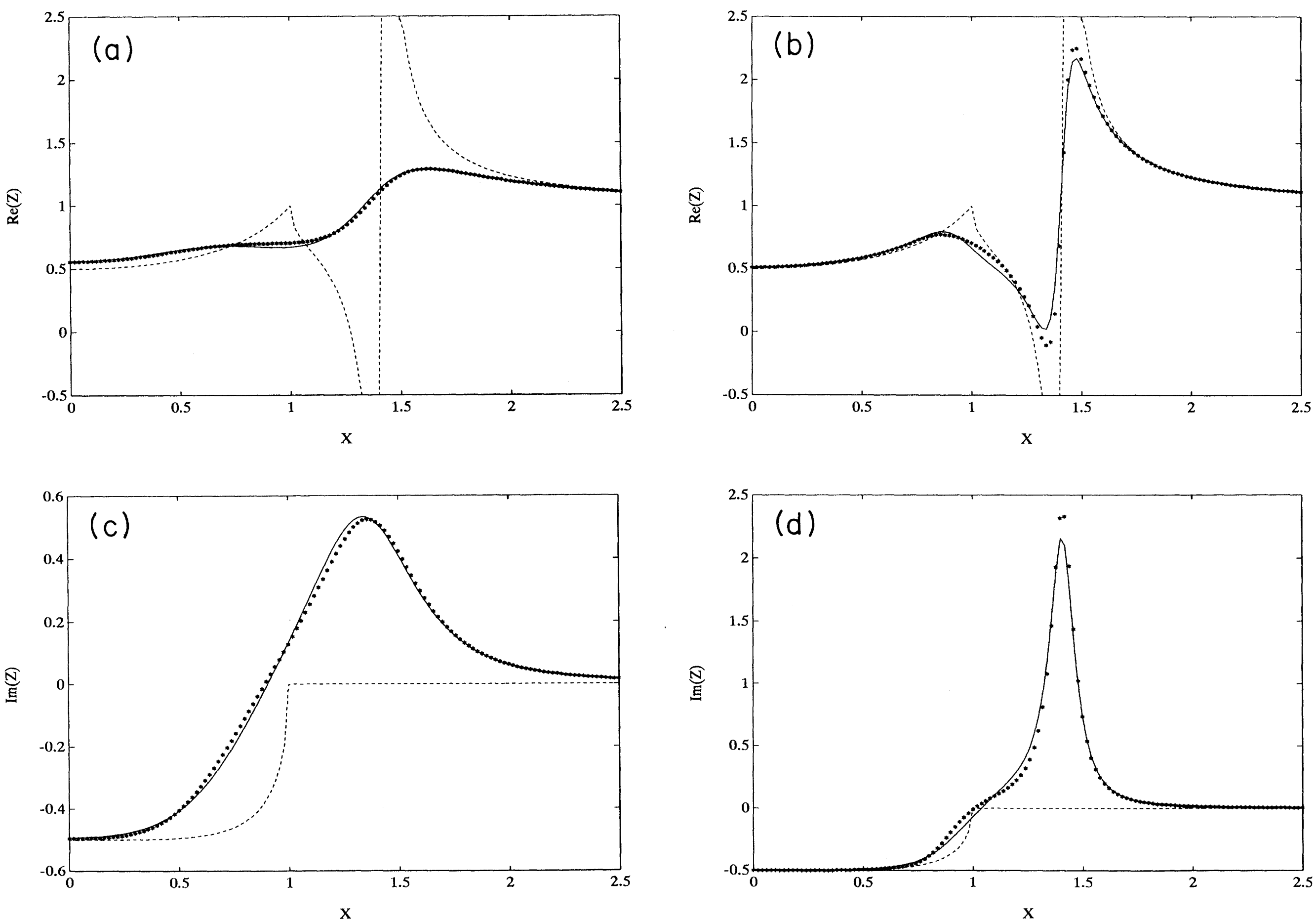

FIG. 5. Comparison of the exact (solid line), approximate (asterisks), and noiseless (dashed line) solutions for the real [(a) $\gamma=10$, (b) $\gamma=50$] and imaginary [(c) $\gamma=10$, (d) $\gamma=50$] parts of the normalized impedance vs x.

$$\tilde{R}_p(\omega)=\frac{\tilde{A}_p(\omega)}{\tilde{A}_{p-1}(\omega)}, \quad p\geq 2 . \tag{A1}$$

We thus have

$$\left[-i\omega\tau_0+p^2+\frac{ip\gamma x}{2}+\frac{\gamma p}{4}\tilde{R}_{p+1}\right]\tilde{R}_p = \frac{\gamma p}{4}-\frac{ip\gamma}{2}\frac{\langle r^p\rangle_0}{\tilde{A}_{p-1}(\omega)} . \tag{A2}$$

We seek a complete solution of the inhomogeneous equation (A2). We can regard Eq. (A2) as having a particular solution and a complementary solution. The particular solution satisfies Eq. (13), i.e.,

$$\left[-i\omega\tau_0+p^2+\frac{ip\gamma x}{2}+\frac{\gamma p}{4}\tilde{S}_{p+1}\right]\tilde{S}_p=\frac{\gamma p}{4} . \tag{A3}$$

Following the methods of Coffey [19] and Cresser *et al.* [11] we write

$$\tilde{R}_p(\omega)=\tilde{S}_p(\omega)+\tilde{Q}_p(\omega) . \tag{A4}$$

Equation (A2) then becomes

$$\left[-i\omega\tau_0+p^2+\frac{ip\gamma x}{2}+\frac{\gamma p}{4}[\tilde{S}_{p+1}+\tilde{Q}_{p+1}]\right](\tilde{S}_p+\tilde{Q}_p) = \frac{\gamma p}{4}-\frac{ip\gamma}{2}\frac{\langle r^2\rangle_0}{\tilde{A}_{p-1}} . \tag{A5}$$

Using Eq. (A3), Eq. (A5) simplifies to

$$\left[-i\omega\tau_0+p^2+\frac{ip\gamma x}{2}\right]\tilde{A}_{p-1}\tilde{Q}_p+\frac{\gamma p}{4}[\tilde{Q}_{p+1}\tilde{S}_p\tilde{A}_{p-1}+\tilde{Q}_p\tilde{A}_{p-1}(\tilde{S}_{p+1}+\tilde{Q}_{p+1})]=-\frac{ip\gamma}{2}\langle r^p\rangle_0 . \tag{A6}$$

Let us write

$$q_p=\tilde{A}_{p-1}\tilde{Q}_p .$$

Then

$$\left[-i\omega\tau_0+p^2+\frac{ip\gamma x}{2}\right]q_p+\frac{\gamma p}{4}[\tilde{Q}_{p+1}\tilde{A}_{p-1}(\tilde{S}_p+\tilde{Q}_p)+q_p\tilde{S}_{p+1}]=-\frac{ip\gamma}{2}\langle r^p\rangle_0 . \tag{A7}$$

Now using Eqs. (A1) and (A4) we have

$$\tilde{R}_p=\tilde{S}_p+\tilde{Q}_p=\frac{\tilde{A}_p}{\tilde{A}_{p-1}} \tag{A8}$$

so that

$$\tilde{R}_p\tilde{A}_{p-1}=(\tilde{S}_p+\tilde{Q}_p)\tilde{A}_{p-1}=\tilde{A}_p . \tag{A9}$$

We can use Eq. (A9) in Eq. (A10) to obtain

$$\left[-i\omega\tau_0+p^2+\frac{ip\gamma x}{2}\right]q_p+\frac{\gamma p}{4}(q_{p+1}+q_p\tilde{S}_{p+1}) = -\frac{ip\gamma}{2}\langle r^p\rangle_0 . \tag{A10}$$

Solving for q_p we get

$$q_p=\frac{-\frac{ip\gamma}{2}\langle r^p\rangle_0-\frac{\gamma p}{4}q_{p+1}}{-i\omega\tau_0+p^2+\frac{ip\gamma x}{2}+\frac{\gamma p}{4}\tilde{S}_{p+1}} . \tag{A11}$$

The complete solution of the inhomogeneous equation is given by Eq. (A8), that is

$$\tilde{A}_p=\tilde{S}_p\tilde{A}_{p-1}+\tilde{Q}_p\tilde{A}_{p-1} \tag{A12}$$

or

$$\tilde{A}_p=\tilde{S}_p\tilde{A}_{p-1}+q_p . \tag{A13}$$

Using the expression for $\tilde{S}_p$ from Eq. (A3) and the expression for q_p from Eq. (A11) in Eq. (A13) we finally get

$$\tilde{A}_p(\omega)=\frac{\frac{\gamma p}{4}\tilde{A}_{p-1}(\omega)-\frac{ip\gamma}{2}\langle r^p\rangle_0-\frac{\gamma p}{4}q_{p+1}}{-i\omega\tau_0+p^2+\frac{ip\gamma x}{2}+\frac{\gamma p}{4}\tilde{S}_{p+1}} . \tag{A14}$$

Using the initial conditions $\tilde{A}_0(\omega)=0$ we can write Eq. (A14) for $p=1$ as follows:

$$\tilde{A}_1(\omega)=\frac{-0.5i\gamma\langle r\rangle_0-0.25\gamma q_2}{-i\omega\tau_0+1+0.5i\gamma x+0.25\tilde{S}_2(\omega)} .$$

We can obtain an expression for q_2 from Eq. (A11). Therefore

$$\tilde{A}_1(\omega)=-2i\tilde{S}_1(\omega)\langle r\rangle_0+\frac{-0.25i\gamma^2\langle r^2\rangle_0+0.125\gamma q_3}{[-i\omega\tau_0+1+0.5i\gamma x+0.25\tilde{S}_2(\omega)][-i\omega\tau_0+4+i\gamma x+0.5\tilde{S}_3(\omega)]}$$

$$=-2i\tilde{S}_1(\omega)\langle r\rangle_0+2i\tilde{S}_1(\omega)\tilde{S}_2(\omega)\langle r^2\rangle_0 \tag{A15}$$

$$+\frac{0.25i\gamma^2 q_3}{[-i\omega\tau_0+1+0.5i\gamma x+0.25\tilde{S}_2(\omega)][-i\omega\tau_0+4+i\gamma x+0.5\tilde{S}_3(\omega)]} . \tag{A16}$$

Substituting for q_3, q_4 and so on in Eq. (A16) we obtain

$$\tilde{A}_1(\omega)=-2i\left[\tilde{S}_1(\omega)\langle r\rangle_0-\tilde{S}_1(\omega)\tilde{S}_2(\omega)\langle r^2\rangle_0+\tilde{S}_1(\omega)\tilde{S}_2(\omega)\tilde{S}_3(\omega)\langle r^3\rangle_0-\cdots(-1)^{n+1}\langle r^n\rangle_0\prod_{k=1}^{n}\tilde{S}_k(\omega)+\cdots\right], \tag{A17}$$

where the $\tilde{S}_n(\omega)$ are given by Eq. (24).

On taking account of Eqs. (26), Eq. (A17) becomes

$$\tilde{A}_1(\omega)=-2i\tilde{S}_1(\omega)\tilde{S}_1(0)\{1-\tilde{S}_2(\omega)\tilde{S}_2(0)\times[1-\tilde{S}_3(\omega)\tilde{S}_3(0)(1-\cdots)]\}\ . \tag{A18}$$

Equation (A18) can further be simplified as follows. Let us introduce a new quantity $\tilde{R}_n(\omega)$ defined as

$$\tilde{R}_n(\omega)=\tilde{S}_n(\omega)\tilde{S}_n(0)[1-\tilde{R}_{n+1}(\omega)]=\tilde{S}_n(\omega)\tilde{S}_n(0)[1-\tilde{S}_{n+1}(\omega)\tilde{S}_{n+1}(0)(1-\cdots)]\ . \tag{A19}$$

The solution of Eq. (A19) is by inspection (as may be seen by direct substitution)

$$\tilde{R}_n(\omega)=\frac{i\gamma n}{4\tau_0}\left[\frac{\tilde{S}_n(0)-\tilde{S}_n(\omega)}{\omega}\right]. \tag{A20}$$

Hence on comparing Eqs. (A18), (A19), and (A20) we have

$$\tilde{A}_1(\omega)=-2i\tilde{R}_1(\omega)=-\frac{\gamma}{2\tau_0}\left[\frac{\tilde{S}_1(\omega)-S_1}{\omega}\right]. \tag{A21}$$

We can show in the same way that the response of $\langle r^{-1}\rangle$, where

$$\langle r^{-1}\rangle=\langle r^{-1}\rangle_0+\tilde{A}_{-1}(\omega)I_m e^{-i\omega t}/I+\cdots$$

is given by

$$\tilde{A}_{-1}(\omega)=2i\tilde{S}_1^*(-\omega)\tilde{S}_1^*(0)(1-\tilde{S}_2^*(-\omega)\tilde{S}_2^*(0)\{1-\tilde{S}_3^*(-\omega)\tilde{S}_3^*(0)[1-\tilde{S}_4^*(-\omega)\tilde{S}_4^*(0)(1-\cdots)]\})=-\frac{\gamma}{2\tau_0}\left[\frac{\tilde{S}_1^*(-\omega)-S_1^*}{\omega}\right], \tag{A22}$$

where the asterisk means the complex conjugate. Equations (A18)–(A22) are very convenient for numerical calculations.

APPENDIX B

In this appendix we evaluate $\partial/\partial\Delta\langle r\rangle_0$ and $\partial/\partial\Delta\langle r^2\rangle_0$. On using Eq. (44) in Eq. (35), we obtain

$$\left[n^2+\frac{in\hbar}{2ekT}(I_{\rm dc}+\Delta)\right]\left[\langle r^n\rangle_0+\Delta\frac{\partial}{\partial\Delta}\langle r^n\rangle_0\right]+\frac{n\hbar I}{4ekT}\left[\langle r^{n+1}\rangle_0+\Delta\frac{\partial}{\partial\Delta}\langle r^{n+1}\rangle_0\right]-\left[\langle r^{n+1}\rangle_0+\Delta\frac{\partial}{\partial\Delta}\langle r^{n+1}\rangle_0\right]=0\ . \tag{B1}$$

Thus the linear approximation in Δ is given by the following set of equations:

$$\left[\frac{2n}{\gamma}+ix\right]\langle r^n\rangle_0+\frac{1}{2}(\langle r^{n+1}\rangle_0-\langle r^{n-1}\rangle_0)=0\ , \tag{B2}$$

$$\left[\frac{2n}{\gamma}+ix\right]\frac{\partial}{\partial\Delta}\langle r^n\rangle_0+\frac{1}{2}\left[\frac{\partial}{\partial\Delta}\langle r^{n+1}\rangle_0-\frac{\partial}{\partial\Delta}\langle r^{n-1}\rangle_0\right]=-\frac{2i}{I}\langle r^n\rangle_0\ . \tag{B3}$$

As we already know the solution of Eq. (B2) may again be given in terms of an infinite continued fraction.

$$\frac{\langle r^n\rangle_0}{\langle r^{n-1}\rangle_0}=\frac{0.5}{\dfrac{2n}{\gamma}+ix+\dfrac{1}{2}\dfrac{\langle r^{n+1}\rangle_0}{\langle r^n\rangle_0}}\ . \tag{B4}$$

On noting that

$$\langle r^0\rangle_0=1$$

we obtain the well known results [6]

$$\langle r\rangle_0=\cfrac{0.5}{\cfrac{2}{\gamma}+ix+\cfrac{0.25}{\cfrac{4}{\gamma}+ix+\cfrac{0.25}{\cfrac{6}{\gamma}+ix+\cfrac{0.25}{\cfrac{8}{\gamma}+ix+\cdots}}}}\ . \tag{B5}$$

The other quantities $\langle r^n\rangle_0$ with $n\geq 2$ can be obtained from the recurrence relation of Eq. (B2) by iteration, for example,

$$\langle r^2\rangle_0=1-2\left[\frac{2}{\gamma}+ix\right]\langle r\rangle_0 \tag{B6}$$

and so on.

On substituting $\langle r^n\rangle_0$ into Eq. (B3) and noting that it has the same form as Eq. (18) we have the solution similar to that of Appendix A:

$$\frac{\partial}{\partial \Delta}\langle r\rangle_0 = -\frac{2i}{I}\left[S_1\langle r\rangle_0 - S_1S_2\langle r^2\rangle_0 + S_1S_2S_3\langle r^3\rangle_0 - \cdots + (-1)^{n-1}\prod_{k=1}^{n} S_k\langle r^n\rangle_0 + \cdots\right], \tag{B7}$$

where

$$S_k = \cfrac{0.5}{\cfrac{2k}{\gamma} + ix + \cfrac{0.25}{\cfrac{2(k+1)}{\gamma} + ix + \cfrac{0.25}{\cfrac{2(k+2)}{\gamma} + ix + \cdots}}}. \tag{B8}$$

Noting that Eq. (26) allows us to express S_n in terms of $\langle r^n\rangle_0$ as

$$\langle r^n\rangle_0 = \langle r^{n-1}\rangle_0 S_n \tag{B9}$$

we obtain

$$\frac{\partial}{\partial \Delta}\langle r\rangle_0 = -\frac{2i}{I}[\langle r\rangle_0^2 - \langle r^2\rangle_0^2 + \langle r^3\rangle_0^2 - \langle r^4\rangle_0^2 + \cdots]$$
$$= \frac{2i}{I}\sum_{n=1}^{\infty}(-1)^n\langle r^n\rangle_0^2 . \tag{B10}$$

The quantity $\partial\langle r^2\rangle_0/\partial\Delta$ can be obtained from Eq. (B3) at $n=1$, where

$$\frac{\partial}{\partial \Delta}\langle r^2\rangle_0 = -\frac{2i}{I}\langle r\rangle_0 - 2\left[\frac{2}{\gamma}+ix\right]\frac{\partial}{\partial \Delta}\langle r\rangle_0$$
$$= -\frac{2i}{I}\langle r\rangle_0 + \frac{4i}{I}\left[\frac{2}{\gamma}+ix\right]$$
$$\times[\langle r\rangle_0^2 - \langle r^2\rangle_0^2 + \langle r^3\rangle_0^2 - \langle r^4\rangle_0^2 + \cdots] . \tag{B11}$$

On substituting Eqs. (B10) and (B11) into Eq. (38) we obtain

$$\lambda_{\rm ef}^{+} = \frac{\gamma}{4\tau_0}\frac{\langle r\rangle_0}{\sum_{n=1}^{\infty}(-1)^{n+1}\langle r^n\rangle_0^2} . \tag{B12}$$

[1] V. Ambegaokar and V. I. Halperin, Phys. Rev. Lett. **22**, 1364 (1969).
[2] Yu. M. Ivanchenko and L. A. Zil'berman, Zh. Eksp. Teor. Fiz. **55**, 2395 (1968) [Sov. Phys. JETP **28**, 1272 (1969)].
[3] M. J. Stephen, Phys. Rev. Lett. **21**, 1629 (1968).
[4] M. J. Stephen, Phys. Rev. **182**, 531 (1969).
[5] W. H. Henkels and W. W. Webb, Phys. Rev. Lett. **26**, 164 (1971).
[6] H. Risken, *The Fokker-Planck Equation* (Springer-Verlag, Berlin, 1984).
[7] G. Barone and A. Paterno, *Physics and Application of the Josephson Effect* (Wiley-Interscience, New York, 1982).
[8] F. Auracher and T. Van Duzer, J. Appl. Phys. **44**, 848 (1973).
[9] C. V. Stancampiano, IEEE Trans. Electron Devices **27**, 1934 (1980).
[10] J. D. Cresser, W. H. Louisell, P. Meystre, W. Schleich, and M. O. Scully, Phys. Rev. A **25**, 2214 (1982).
[11] J. D. Cresser, D. Hammonds, W. H. Louisell, P. Meystre, and H. Risken, Phys. Rev. A **25**, 2226 (1982).
[12] J. D. Cresser, Phys. Rev. A **26**, 398 (1982).
[13] H. Haken, H. Sauermann, C. Schmidt, and H. D. Vollmer, Z. Phys. **206**, 269 (1967).
[14] W. W. Chow, M. O. Scully, and E. W. Van Stryland, Opt. Commun. **15**, 6 (1975).
[15] A. J. Viterbi, Proc. IEEE **51**, 1737 (1963).
[16] K. K. Likharev and V. K. Semenov, Radiotekh. Elektron. **18**, 1767 (1973).
[17] A. N. Vystavkin, V. N. Gubankov, L. S. Kuzmin, K. K. Likharev, V. V. Migulin, and V. K. Semenov, Rev. Phys. Appl. **9**, 79 (1974).
[18] W. T. Coffey, Yu. P. Kalmykov, and E. S. Massawe, in *Modern Non-Linear Optics,* edited by I. Prigogine, S. A. Rice, and M. W. Evans [Adv. Chem. Phys. (to be published)].
[19] W. T. Coffey, in *Dynamical Processes in Condensed Matter,* edited by M. W. Evans (Wiley-Interscience, New York, 1985), Vol. 63, p. 69.
[20] H. Fröhlich, *Theory of Dielectrics,* 2nd ed. (Oxford Univ. Press, Oxford, 1958).